Ruin probabilities with risky investments

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The (ultimate) ruin problem in great generality can be formulated as follows. Let (P, R) be a two-dimensional semimartingale such that $\Delta R > -1$ and let $X = X^u$ be the solution of linear stochastic equation

$$dX_t = X_{t-}dR_t + dP_t, \quad X_0 = u > 0.$$

Find the function $\Psi(u) := P(\tau^u < \infty)$ where $\tau^u := \inf\{t : X_t^u \le 0\}$.

In the financial-actuarial context the process P serves to describe the capital reserve of an insurance company, $\mathcal{E}(R)$ is a price process, $\Psi(u)$ is the ruin probability. In the classical case (no investments, zero interest rate), $R = 0$ and P is a compound Poisson process with drift (the Lundberg–Cramér) or a renewal process with drift (the Sparre Andersen model). Of course, the ruin probability can be found in an explicit form in rare cases and the majority of studies deal with asymptotic of the function $\Psi(u)$ as $u \to \infty$ assuming that P and R are independent Lévy processes. It happens that for nontrivial R the asymptotic behavior of the ruin probability is radically different from those in the classical models where it decays exponentially. E.g., if the price dynamics is given by a geometric Brownian motion (i.e. $dR_t = adt + \sigma dW_t$, $\sigma \ne 0$) and P is a compound Poisson process with drift and exponentially distributed jumps, one can use methods of integro-differential equations and get that $\Psi(u) \sim Cu^{-\beta}$, $C > 0$, when $\beta := 2a/\sigma^2 - 1 > 0$ while in the case where $\beta \le 0$ the ruin is imminent. Similar results were obtained recently also for the Sparre Andersen models with investments.

For the model where P and R are independent Lévy processes introduced by Paulsen in 1973 and studied in the series of his papers it was known, under some assumptions, that the ruin probability $\Psi(u) \sim Cu^{-\beta}$, where $\beta > 0$ is the root of the equation $H(\beta) = 0$ assumed to existing. Here H is the cumulant generating function of the increment of the log price process $V_t = \ln \mathcal{E}_t(R)$ of the risky asset, that is, $H(p) := \ln \mathbf{E}e^{-qV_1}$. To find the asymptotic one can use Kesten–Coldie implicit renewal theory (known also as the theory of random equations or equations in sense of laws). Unfortunately, until recently, this theory did not give a direct answer to the question whether the constant C is strictly positive. Proofs of this important property led to additional assumptions and cumbersome formulations. A recent progress in the implicit renewal theory allows to get the following short formulation.

Theorem 1 *Suppose that the law of V_1 is not arithmetic, the process P has jumps only upward, and $H(\beta+) < \infty$. If $\int |x|^\beta I_{\{x>1\}}\Pi_P(dx) < \infty$, where Π_P is the Lévy measure of the process P, then $\Psi(u) \sim Cu^{-\beta}$ with $C > 0$.*